CONJUGACY AND THE $3x + 1$ CONJECTURE

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1. Introduction

The $3x + 1$ conjecture involves the iteration of the function $T : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{3x+1}{2} & \text{if } x \text{ is odd.} \end{cases}$$

The conjecture states that for every positive integer x, there exists a positive integer k such that $T^k(x) = 1$ where T^k is the k-fold composition of T with itself. The function T may be extended to the 2-adic integers $\mathbb{Z}_2$ (see section 3) in a natural manner, and for the remainder of the paper we will consider T to be a map on $\mathbb{Z}_2$.

Here we will attempt to investigate the behavior of T by studying the dynamics of functions which are topologically conjugate to T. If $f : X \to X$ and $g : Y \to Y$ where X and Y are topological spaces then f is *conjugate* to g if there exists a bijection $h : X \to Y$ such that $h \circ f = g \circ h$. The map h is called a *conjugacy.* If the function h is also a homeomorphism then we say that f and g are *topologically conjugate*, and we call h a *topological conjugacy.* One property of conjugacy is that it preserves the dynamics of a function. So, if we find a function, f, which is conjugate to T by a relatively simple bijection then we will be able to understand the behavior of T by describing the behavior of f, and thus hopefully answer the conjecture. To this end, we find a family, $\mathcal{F}$, of functions on the 2-adics whose elements are topologically conjugate to T. This family contains all functions which are conjugate to T by linear maps –certainly the simplest of homeomorphisms.

2. **Summary of Results**

In this section we state our main theorems, leaving their proofs for later in the paper. We begin with the following definition.

Definition. A function $f_{a,b,c,d} : \mathbb{Z}_2 \to \mathbb{Z}_2$ is *modular* if it is of the form:

$$f_{a,b,c,d}(x) = \begin{cases} \frac{ax+b}{2} & \text{if } x \text{ is even} \\ \frac{cx+d}{2} & \text{if } x \text{ is odd} \end{cases}$$

with $a, b, c, d \in \mathbb{Z}_2$.

We should note that $f_{a,b,c,d}$ does not define a function for every a, b, c and d. However, the definition of modular requires that $f_{a,b,c,d}$ be a function.

We now define an infinite family of modular functions.

Definition. Let $\mathcal{F}$ be the set of modular functions, $f_{a,b,c,d}$, such that a, c and d are odd and b is even.

Example 1. *Since $T = f_{1,0,3,1}$ we see that $T \in \mathcal{F}$.*

Example 2. *The shift map, σ, is $f_{1,0,1,-1}$, and thus is a member of $\mathcal{F}$ as well.*

The importance of this family is illustrated by the following theorems.

Theorem 1. *Let $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ be a modular function. Then f is conjugate to T if and only if $f \in \mathcal{F}$. Furthermore, every $f \in \mathcal{F}$ is topologically conjugate to T.*

In order to use these maps to study the conjecture we also need to understand the behavior of the conjugacies between the maps and T. To be a conjugacy, a function must be bijective, and one simple type of bijective function is the linear map. We have the following result.

Theorem 2. *Every map which is conjugate to T by a linear homeomorphism is a member of $\mathcal{F}$. In fact, a map is conjugate to T by a linear homeomorphism if and only if it is of the form $f_{1,q,3,p-q}$ where p is odd and q is even or $f_{3,p-q,1,q}$ where both p and q are odd.*

As we will see, the shift map is not conjugate to T by a linear conjugacy. Thus not all elements of $\mathcal{F}$ are conjugate to T by linear maps. Additionally, there are maps which are conjugate to T that are not members of $\mathcal{F}$. Specifically, we will show that a map which is conjugate to T by a piecewise linear function is not necessarily in $\mathcal{F}$.

The parity vector function (see section 3) has played an important part in work done in the past on the $3x+1$ problem ([3], [1]). In order to prove Theorems 1 and 2 above we generalize several of the technical results concerning the properties of the parity vector function to show that they apply to all elements of $\mathcal{F}$ in general, and not just to T. These generalizations are described in Section 4.

The significance of these results may be seen in the following example. Let $F = f_{1,0,1,1}$. It can easily be shown that $F^n(x) = \frac{1}{3}$ and $F^{n+1}(x) = \frac{2}{3}$ for some $n \in \mathbb{Z}^+$ if and only if $x = \frac{a}{3}$ for some integer a which is not divisible by three. In addition, if Φ is the conjugacy between T and F then $\Phi(1) = \frac{1}{3}$ or $\Phi(1) = \frac{2}{3}$ (since $\frac{1}{3}, \frac{2}{3}$ and $1, 2$ the only 2-cycles of F and T respectively). Thus the Collatz conjecture is equivalent to the statement that the Φ image of any positive integer is of the form $\frac{a}{3}$ for some integer a not divisible by three.

3. **Background and Notation**

A 2-adic integer is an infinite sequence $s_0, s_1, s_2, \ldots$ where $s_i \in \{0,1\}$ for all $i \geq 0$. We may consider $\mathbb{Z}^+$ to be a subset of $\mathbb{Z}_2$ by identifying $n \in \mathbb{Z}^+$ with its base-2 expansion written in reverse order and completed with an infinite string of zeros. For clarity we will often write an integer in place of its 2-adic representation.

Addition and multiplication are easily defined on the 2-adics by extending the usual algorithms for adding and multiplying integers in base 2 (see [1] for a nice exposition). With these operations, $\mathbb{Z}_2$ becomes a ring containing $\mathbb{Z}$ as a subring. Just as in $\mathbb{Z}$, $s \in \mathbb{Z}_2$ is odd or even based on its equivalence in $\mathbb{Z}_2/2\mathbb{Z}_2$. Since even and odd are well defined on $\mathbb{Z}_2$ the map T extends nicely.

It is also well known that $\mathbb{Z}_2$ contains the ring of rational numbers with odd denominators. Note that 2 has no multiplicative inverse in $\mathbb{Z}_2$, and therefore $\mathbb{Z}_2$ does not contain the reciprocal of any even integer.

The *parity vector function of length k associated with T*, Q_k, is given by the sequence

$$Q_k(\alpha) = x_0(\alpha), x_1(\alpha), x_2(\alpha), ..., x_{k-1}(\alpha)$$

where $\alpha \in \mathbb{Z}_2$, $x_i(\alpha) \in \{0, 1\}$ and

$$x_i(\alpha) \equiv T^i(\alpha) \bmod 2$$

for all $0 \leq i \leq k - 1$.

We define the *parity vector function associated with T*, Q_∞, similarly. We refer to this function as simply Q.

Lagarias shows that Q_k is periodic with period 2^k, and uses this to prove that Q is a homeomorphism [3].

Another important function is the *shift map*, $\sigma : \mathbb{Z}_2 \to \mathbb{Z}_2$ by

$$\sigma(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x-1}{2} & \text{if } x \text{ is odd.} \end{cases}$$

This function simply "removes" the first digit of a 2-adic integer. That is, $\sigma(s_0, s_1, s_2, ...) = s_1, s_2, ...$ Interestingly enough, this function is chaotic. This is important because T has been proven to be topologically conjugate to σ using Q [3], and hence T is chaotic.

4. Generalization of parity vector functions

Recall $\mathcal{F}$ is the set of functions

$$\{f_{a,b,c,d} : \mathbb{Z}_2 \to \mathbb{Z}_2 | a,b,c,d \in \mathbb{Z}_2 \text{ where } a, c \text{ and } d \text{ are odd, and } b \text{ is even}\}$$

where

$$f_{a,b,c,d}(x) = \begin{cases} \frac{ax+b}{2} & \text{if } x \text{ is even} \\ \frac{cx+d}{2} & \text{if } x \text{ is odd.} \end{cases}$$

We now generalize the notion of a parity vector function for elements of $\mathcal{F}$. Let $f : \mathbb{Z}_2 \to \mathbb{Z}_2$. The *parity vector function of length k associated with f*, Φ_k, is given by the sequence

$$\Phi_k(\alpha) = x_0(\alpha), x_1(\alpha), x_2(\alpha), ..., x_{k-1}(\alpha)$$

where $\alpha \in \mathbb{Z}_2$, $x_i(\alpha) \in \{0, 1\}$ and

$$x_i(\alpha) \equiv f^i(\alpha) \bmod 2$$

for all $0 \leq i \leq k-1$.

We may define the *parity vector function associated with f*, Φ_∞, similarly. We refer to this function as simply Φ.

We will continue to refer to the parity vector function associated with T as Q, and the parity vector function of length k associated with T as Q_k.

5. Technical Results and Proofs

5.1. All members of $\mathcal{F}$ are conjugate to T.

We will begin by showing that the parity vector functions, Φ, retain the important properties of Q. Specifically, we will show that Φ_k is periodic with period 2^k, and that Φ is a homeomorphism. We will then use this to show that elements of the set $\mathcal{F}$ are topologically conjugate to T.

Theorem 3. *Let $f \in \mathcal{F}$ and let Φ_k be the parity vector function of length k associated with f. Then for any positive integer k, Φ_k is periodic with period 2^k.*

We use two lemmas in this proof.

Lemma 1. *Let $f \in \mathcal{F}$ then for any non-negative integer k, $f^k(\alpha+\omega 2^k) \equiv f^k(\alpha)+\omega \bmod 2$, for any $\alpha, \omega \in \mathbb{Z}_2$.*

Proof. Let $f \in \mathcal{F}$ then $f = f_{a,b,c,d}$ for some $a, b, c, d \in \mathbb{Z}_2$, with a, c, d odd and b even. Then let $\alpha, \omega \in \mathbb{Z}_2$. We will use induction on k.

Base Case: Let $k = 0$. Then

$$\begin{aligned} f^0(\alpha+\omega) &= \alpha+\omega \\ &\equiv f^0(\alpha)+\omega \bmod 2. \end{aligned}$$

Inductive Hypothesis: Assume $f^{k-1}(\alpha+\omega 2^{k-1}) \equiv f^{k-1}(\alpha)+\omega \bmod 2$.

We now have two cases depending on the parity of α.

Case 1: α is even.

$$\begin{aligned} f^k(\alpha+\omega 2^k) &= f^{k-1}(f(\alpha+\omega 2^k)) \\ &= f^{k-1}\left(\tfrac{a(\alpha+\omega 2^k)+b}{2}\right) && \text{(since } \alpha \text{ is even)} \\ &= f^{k-1}\left(\tfrac{a\alpha+b}{2}+a\omega 2^{k-1}\right) \\ &\equiv f^{k-1}\left(\tfrac{a\alpha+b}{2}\right)+a\omega \bmod 2 && \text{(by ind. hyp.)} \\ &\equiv f^{k-1}\left(\tfrac{a\alpha+b}{2}\right)+\omega \bmod 2 && \text{(since } a \text{ is odd)} \\ &\equiv f^{k-1}(f(\alpha))+\omega \bmod 2 && \text{(since } \alpha \text{ is even)} \\ &\equiv f^k(\alpha)+\omega \bmod 2 \end{aligned}$$

Case 2: α is odd.

$$\begin{aligned} f^k(\alpha+\omega 2^k) &= f^{k-1}(f(\alpha+\omega 2^k)) \\ &= f^{k-1}\left(\tfrac{c(\alpha+\omega 2^k)+d}{2}\right) && \text{(since } \alpha \text{ is odd)} \\ &= f^{k-1}\left(\tfrac{c\alpha+d}{2}+c\omega 2^{k-1}\right) \\ &\equiv f^{k-1}\left(\tfrac{c\alpha+d}{2}\right)+c\omega \bmod 2 && \text{(by ind. hyp.)} \\ &\equiv f^{k-1}\left(\tfrac{c\alpha+d}{2}\right)+\omega \bmod 2 && \text{(since } c \text{ is odd)} \\ &\equiv f^{k-1}(f(\alpha))+\omega \bmod 2 && \text{(since } \alpha \text{ is odd)} \\ &\equiv f^k(\alpha)+\omega \bmod 2 \end{aligned}$$

Thus, $f^k(\alpha+\omega 2^k) \equiv f^k(\alpha)+\omega \bmod 2$ for all positive integers k. □

Now we produce a similar result concerning the elements of a parity vector.

Lemma 2. *Let $f \in \mathcal{F}$ and let $x_{k-1}(\alpha)$ be the k^{th} term in the parity vector function associated with f. Then for every $\alpha, \omega \in \mathbb{Z}_2$, $x_k(\alpha+\omega 2^k) \equiv x_k(\alpha)+\omega \bmod 2$ for all $0 \le k \le \infty$.*

Proof. Let $f \in \mathcal{F}$. Then let α, $\omega \in \mathbb{Z}_2$ and let $x_{k-1}(\alpha)$ be the k^{th} term in the parity vector function associated with f. Then for all $0 \leq k \leq \infty$ we have the following.

$$\begin{aligned} x_k(\alpha + \omega 2^k) &\equiv f^k(\alpha + \omega 2^k) \bmod 2 \\ &\equiv f^k(\alpha) + \omega \bmod 2 \qquad \text{(by Lemma 1)} \\ &\equiv x_k(\alpha) + \omega \bmod 2 \end{aligned}$$

□

We are now ready to prove Theorem 3.

Proof of Theorem 3. Let $f \in \mathcal{F}$, let Φ_k be the parity vector function of length k associated with f, and let $\alpha, \omega \in \mathbb{Z}_2$. Again, we will use induction on k.

Base Case: Let $k = 1$. Then

$$\begin{aligned} \Phi_1(\alpha + 2\omega) &= x_0(\alpha + 2\omega) \\ &= x_0(\alpha) \qquad \text{(by Lemma 2)} \\ &= \Phi_1(\alpha). \end{aligned}$$

Inductive Hypothesis: Assume $\Phi_{k-1}(\alpha + \omega 2^{k-1}) = \Phi_{k-1}(\alpha)$. Then we have

$$\begin{aligned} \Phi_k(\alpha + \omega 2^k) &= x_0(\alpha + \omega 2^k), x_1(\alpha + \omega 2^k), \ldots, x_{k-1}(\alpha + \omega 2^k) \\ &= \Phi_{k-1}(\alpha + \omega 2^k), x_{k-1}(\alpha + \omega 2^k) \\ &= \Phi_{k-1}(\alpha + \omega 2^k), x_{k-1}(\alpha) \qquad \text{(by Lemma 2)} \\ &= \Phi_{k-1}(\alpha), x_{k-1}(\alpha) \qquad \text{(by ind. hyp.)} \\ &= x_0(\alpha), x_1(\alpha), \ldots, x_{k-1}(\alpha) \\ &= \Phi_k(\alpha). \end{aligned}$$

Thus Φ_k is periodic with period 2^k. □

Now, to show that Φ is a homeomorphism we use a result whose proof is mentioned in [3] for the case $\Phi = Q$. For a more detailed exposition see [1]. Their proofs for Q carry over exactly for any Φ.

Theorem 4. *([3],[1]) If $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ is a function whose parity vector functions of length k, Φ_k, are periodic with period 2^k and for which $x_k(\alpha + \omega 2^k) \equiv x_k(\alpha) + \omega \bmod 2$ for all k, then the parity vector function associated with f, Φ, is a measure preserving homeomorphism.*

So, using this result, Theorem 3 and Lemma 2, we conclude that for any $f \in \mathcal{F}$, the parity vector function associated with f is a homeomorphism. Now recall Theorem 1.

Theorem 1. *Let $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ be a modular function. Then f is conjugate to T if and only if $f \in \mathcal{F}$. Furthermore, every $f \in \mathcal{F}$ is topologically conjugate to T.*

Again, we use several lemmas to show this result.

Lemma 3. *Let $f \in \mathcal{F}$ then f is topologically conjugate to the shift map σ.*

Proof. Let $f \in \mathcal{F}$ and let Φ be the parity vector function associated with f. Theorem 4 tells us that Φ is a homeomorphism. Then for any $\alpha \in \mathbb{Z}_2$ we have

$$\begin{aligned}
(\Phi \circ f)(\alpha) &= \Phi(f(\alpha)) \\
&= x_0(f(\alpha)), x_1(f(\alpha)), x_2(f(\alpha)), \ldots \\
&= x_1(\alpha), x_2(\alpha), x_3(\alpha), \ldots \\
&= \sigma(x_0(\alpha), x_1(\alpha), x_2(\alpha), \ldots) \\
&= \sigma(\Phi(\alpha)) \\
&= (\sigma \circ \Phi)(\alpha).
\end{aligned}$$

Thus, f is topologically conjugate to σ. □

Now, since topological conjugacy is an equivalence relation, clearly for $f, g \in \mathcal{F}$, f is topologically conjugate to g. In particular, for any $f \in \mathcal{F}$, f is topologically conjugate to T. We should note that for $f, g \in \mathcal{F}$ with parity vector functions Φ and Ψ respectively, the conjugacy between f and g is the homeomorphism $\Psi^{-1} \circ \Phi$.

Lemma 4. *Let $a, b, x, y \in \mathbb{Z}_2$ where $a \neq 0$. If $ax + b = ay + b$ then $x = y$.*

Proof.

$$\begin{aligned}
ax + b = ay + b &\Rightarrow ax = ay \\
&\Rightarrow ax - ay = 0 \\
&\Rightarrow a(x - y) = 0
\end{aligned}$$

It is well known (e.g. [3]) that $\mathbb{Z}_2$ is an integral domain. Since $a \neq 0$ we must have $(x-y) = 0$, and thus $x = y$. □

Lemma 5. *Let $f : \mathbb{Z}_2 \to \mathbb{Z}_2$. If f is conjugate to T and modular then $f \in \mathcal{F}$.*

Proof. Let $f_{a,b,c,d}$ be a modular function and assume that $f_{a,b,c,d}$ is conjugate to T. Then $f_{a,b,c,d}$ must have the same dynamics as σ since σ is conjugate to T. In particular, every number in $\mathbb{Z}_2$ has exactly two distinct preimages under σ, so the same must hold for $f_{a,b,c,d}$.

First, we note that if $a = 0$ then $\frac{b}{2}$ has infinitely many preimages, contradicting the assumption that f is conjugate to σ. Similarly, if $c = 0$ then $\frac{d}{2}$ has infinitely many preimages, again contradicting the assumption that f is conjugate to σ. Thus $a, c \neq 0$.

Now, in order for a point n to have two preimages, we must have $f_{a,b,c,d}(x) = n$ and $f_{a,b,c,d}(y) = n$ for some $x, y \in \mathbb{Z}_2$. Note that since $a \neq 0$, $\frac{as+b}{2} = \frac{at+b}{2} \Rightarrow as+b = at+b \Rightarrow s = t$ by Lemma 4. Similarly, since $c \neq 0$, $\frac{cs+d}{2} = \frac{ct+d}{2} \Rightarrow cs + d = ct + d \Rightarrow s = t$ by Lemma 4. Therefore, one of the x or y must be even, and the other must be odd. Without loss of generality assume x is even and y is odd.

Since x is even we note that $n = \frac{ax+b}{2}$ and then $b = 2n - ax$. We may conclude from this equation that b is even.

For any n we must find an x such that $ax = 2n - b$. If we let $n = 1 + \frac{b}{2}$ then we have $ax = 2(1+\frac{b}{2})-b = 2$. Then $ax = 2$, and since x is even this implies that a is odd. Therefore, if $f_{a,b,c,d}$ is conjugate to T then a is odd.

Now, since y is odd we note that $n = \frac{cy+d}{2}$ then $cy = 2n - d$. We conclude from this equation that $c \equiv d \bmod 2$.

Assume c and d are even, and let $n = c + \frac{d}{2}$. Then $cy = 2(c + \frac{d}{2}) - d = 2c$. Now, since $c \neq 0$, and since $\mathbb{Z}_2$ is an integral domain (see [3]) we conclude that $y = 2$. However, this contradicts our assumption that y is odd. Therefore, if $f_{a,b,c,d}$ is conjugate to T then c and d are odd.

Thus, if $f_{a,b,c,d}$ is conjugate to T then $f_{a,b,c,d} \in \mathcal{F}$. □

Proof of Theorem 1. Let $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ be a modular function. If f is conjugate to T then by Lemma 5, $f \in \mathcal{F}$.

Now, as noted, if $f \in \mathcal{F}$ then by Lemma 3, f is topologically conjugate to T. However, whenever two functions are topologically conjugate they are also conjugate. Thus f is conjugate to T. □

5.2. **Some Simple Conjugacies.** Now that we have established that our family $\mathcal{F}$ contains all the modular functions which are topologically conjugate to T we proceed to show that some of these functions are conjugate by simple, namely linear, homeomorphisms.

5.2.1. *Linear Conjugacies.* We will begin with linear conjugacies. It is easy to check that linear bijections with 2-adic coefficients are continuous with linear inverses, and thus homeomorphisms. Therefore, all linear conjugacies are also topological conjugacies.

First we note that for a linear function $G(x) = px + q$ with 2-adic coefficients to be a bijection it is necessary and sufficient to have p is odd. This result follows simply by letting $p = r$ and $q = s$ in Theorem 5 (see section 5.2.2).

Recall Theorem 2.

Theorem 2. *Every map which is conjugate to T by a linear homeomorphism is a member of $\mathcal{F}$. In fact, a map is conjugate to T by a linear homeomorphism if and only if it is of the form $f_{1,q,3,p-q}$ where p is odd and q is even or $f_{3,p-q,1,q}$ where both p and q are odd.*

Proof. Let $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ with $G(x) = px + q$ a linear conjugacy between T and f where $p, q \in \mathbb{Z}_2$. Then, $f(x) = (G \circ T \circ G^{-1})(x)$ and G is a homeomorphism. Since G is a homeomorphism, it is a bijection, and then as noted above, we must have p odd.

We will compute $(G \circ T \circ G^{-1})(x)$.

$$G^{-1}(x) = \frac{x - q}{p}$$

$$(T \circ G^{-1})(x) = \begin{cases} \frac{x-q}{2p} & \text{if } x-q \text{ is even} \\ \frac{3x-3q+p}{2p} & \text{if } x-q \text{ is odd} \end{cases}$$

$$(G \circ T \circ G^{-1})(x) = \begin{cases} \frac{x+q}{2} & \text{if } x-q \text{ is even} \\ \frac{3x+p-q}{2} & \text{if } x-q \text{ is odd} \end{cases}$$

Now we note that when q is even we have $x-q$ is even if and only if x is even and $x-q$ is odd if and only if x is odd. If q is odd just the opposite is true. Now f is given by the following cases.

Case 1: q is even.

$$\begin{aligned} f(x) &= (G \circ T \circ G^{-1})(x) = \begin{cases} \frac{x+q}{2} & \text{if } x \text{ is even} \\ \frac{3x+p-q}{2} & \text{if } x \text{ is odd} \end{cases} \\ &= f_{1,q,3,p-q}(x) \end{aligned}$$

Case 2: q is odd.

$$\begin{aligned} f(x) &= (G \circ T \circ G^{-1})(x) = \begin{cases} \frac{3x+p-q}{2} & \text{if } x \text{ is even} \\ \frac{x+q}{2} & \text{if } x \text{ is odd} \end{cases} \\ &= f_{3,p-q,1,q}(x) \end{aligned}$$

Thus, any function topologically conjugate to T by a linear map is of the form $f_{1,q,3,p-q}$ where p is odd and q is even, or of the form $f_{3,p-q,1,q}$ where p is odd and q is odd. □

Even though all functions conjugate to T by a linear conjugacy belong to $\mathcal{F}$, not all functions in $\mathcal{F}$ are conjugate to T by a linear conjugacy. For instance $\sigma = f_{1,0,1,-1}$ is not of either form stated in Theorem 2. Therefore, the conjugacy between T and σ, Q, must be a nonlinear function (this can also be proven quite easily by simply computing the images of 3 values under Q and noting that they are not collinear).

This result is actually somewhat unfortunate since σ has very easy to understand behavior, and if we could compute the image of $\mathbb{Z}^+$ under Q we would certainly be able to say much about T. Unfortunately the behavior of Q is very complex and anything but linear.

5.2.2. *Piecewise Linear Conjugacies.* Finally, we look at piecewise linear conjugacies. First we describe what conditions are necessary and sufficient for a piecewise linear function to be bijective.

Theorem 5. *Let $p, q, r, s \in \mathbb{Z}_2$, and let $G : \mathbb{Z}_2 \to \mathbb{Z}_2$ by*

$$G(x) = \begin{cases} px + q & \text{if } x \text{ is even} \\ rx + s & \text{if } x \text{ is odd.} \end{cases}$$

Then G is a bijection if and only if p and r are odd and $q \equiv s \bmod 2$

We need a lemma before proving this theorem.

Lemma 6. *Let $G : \mathbb{Z}_2 \to \mathbb{Z}_2$ as above. If G is onto then*

$$\begin{array}{lll} G([0]) = [0] & \text{and} & G([1]) = [1] \quad \text{or} \\ G([0]) = [1] & \text{and} & G([1]) = [0]. \end{array}$$

That is, either G maps evens to evens and odds to odds, or else G maps evens to odds and odds to evens.

Proof. Let G be onto, and $x \in \mathbb{Z}_2$.

If x is even, then $px \equiv 0 \bmod 2$. So $G(x) = px + q \equiv q \bmod 2$ and thus $G([0]) \subseteq [q]$. Since q is either even or odd, we have $G([0]) \subseteq [0]$, or $G([0]) \subseteq [1]$.

If x is odd, then $rx \equiv r \bmod 2$, so $G(x) = rx + s \equiv r + s \bmod 2$ and thus $G([1]) \subseteq [r + s]$. Since $r + s$ is either even or odd, we have $G([1]) \subseteq [1]$, or $G([1]) \subseteq [0]$.

If $G([0]) \subseteq [0]$ then $G([1]) \subseteq [1]$ since G is onto and in fact, $G([0]) = [0]$ and $G([1]) = [1]$. Similarly if $G([0]) \subseteq [1]$ then $G([1]) \subseteq [0]$ since G is onto and in fact, $G([0]) = [1]$ and $G([1]) = [0]$. □

Now we are ready to prove Theorem 5.

Proof of Theorem 5. Let $G : \mathbb{Z}_2 \to \mathbb{Z}_2$ as stated in Theorem 5.

(⇒) We will begin by showing that if G is a bijection then p and r are odd and $q \equiv s \bmod 2$.

Note that if $p=0$ then $G(0)=G(2)$ which contradicts the assumption that G is bijective. Similarly, if $r=0$ then $G(1)=G(3)$ contradicting the assumption that G is bijective. Thus $p,r\neq 0$.

First, assume p is not odd, then we have two cases.

Case 1: p is even and q is even.

$G(0)=q$ is even and so $G([0])=[0]$ by Lemma 6. Since $p+q$ is even there exists an even x such that $G(x)=p+q$. Then $px+q=p+q$ and since $p\neq 0$ Lemma 4 implies that $x=1$ which is odd, contradicting our assumption about x.

Case 2: p is even and q is odd.

$G(0)=q$ is odd and so $G([0])=[1]$ by Lemma 6. Since $p+q$ is odd there exists an even x such that $G(x)=p+q$. Then $px+q=p+q$ and since $p\neq 0$ Lemma 4 implies that $x=1$ which is odd, contradicting our assumption about x.

Therefore whenever G is a bijection p is odd.

Now, assume r is not odd, then we have two more cases.

Case 3: r is even and s is even.

$G(1)=r+s$ is even and so $G([1])=[0]$ by Lemma 6. Then since $2r+s$ is even there exists an odd x such that $G(x)=2r+s$. Thus $rx+s=2r+s$ and since $r\neq 0$ Lemma 4 implies that $x=2$ which is even, contradicting our assumption about x.

Case 4: r is even and s is odd.

$G(1)=r+s$ is odd and so $G([1])=[1]$ by Lemma 6. Then since $2r+s$ is odd there exists an odd x such that $G(x)=2r+s$. Thus $rx+s=2r+s$ and since $r\neq 0$ Lemma 4 implies that $x=2$ which is even, contradicting our assumption about x.

Therefore whenever G is a bijection r is odd.

Now we will show that if G is a bijection then $q\equiv s \bmod 2$

Case 1: Let q be even. Then $G(0) = q$ is even which implies that $G([0]) = [0]$, and $G([1]) = [1]$ by Lemma 6. Then $G(1) = r + s$ is odd and since r is odd s must be even. So $q \equiv s \bmod 2$.

Case 2: Now let q be odd. Then $G(0) = q$ is odd which implies that $G([0]) = [1]$, and $G([1]) = [0]$ by Lemma 6. Then $G(1) = r + s$ is even and since r is odd s must be odd. So $q \equiv s \bmod 2$.

Therefore when G is a bijection, $q \equiv s \bmod 2$

($\Leftarrow$) Now we will show that if p and r are odd and $q \equiv s \bmod 2$ then G is a bijection.

Assume p and r are odd and $q \equiv s \bmod 2$. Note that since p and r are odd $p, r \neq 0$.

First we will show that G is injective. Let $x, y \in \mathbb{Z}_2$ and let $G(x) = G(y)$. Now we have several cases to consider.

Case 1: When x is even and y is even we have $px + q = py + q$. Thus $x = y$ by Lemma 4.

Case 2: When x is even, and y is odd we have

$$px + q = ry + s \Rightarrow px \equiv ry \bmod 2 \Rightarrow x \equiv y \bmod 2$$

which is a contradiction. Therefore, we cannot have an even x and an odd y.

Case 3: When x is odd and y is even we have

$$rx + s = py + q \Rightarrow rx \equiv py \bmod 2 \Rightarrow x \equiv y \bmod 2$$

which is a contradiction. Therefore, we cannot have an odd x and an even y.

Case 4: When x is odd and y is odd we have $rx + s = ry + s$. Thus $x = y$ by Lemma 4.

Therefore, $G(x) = G(y)$ implies $x = y$, so G is injective.

Now we show that G is a surjection.

Let $y \in \mathbb{Z}_2$. If y is odd and q and s are odd then, $\frac{y-q}{p}$ is even, and $G(\frac{y-q}{p}) = y$.

If y is odd and q and s are even then, $\frac{y-s}{r}$ is odd, and $G(\frac{y-s}{r}) = y$

If y is even and q and s are odd then, $\frac{y-s}{r}$ is odd, and $G(\frac{y-s}{r}) = y$.

If y is even and q and s are even then, $\frac{y-q}{p}$ is even, and $G(\frac{y-q}{p}) = y$

Therefore, for any $y \in \mathbb{Z}_2$ there exists an x such that $G(x) = y$, and so G is a surjection.

Thus G is a bijection. □

Now we will derive the general form of a map which is conjugate to T by a piecewise linear map in order to see if any of these are not members of $\mathcal{F}$.

Let $p, q, r, s \in \mathbb{Z}_2$ where p and r are odd, $q \equiv s \bmod 2$ and $G : \mathbb{Z}_2 \to \mathbb{Z}_2$ by

$$G(x) = \begin{cases} px + q & \text{if } x \text{ is even} \\ rx + s & \text{if } x \text{ is odd.} \end{cases}$$

Then by Theorem 5 G is a bijection. Let $f = (G \circ T \circ G^{-1})$. We wish to compute $f(x)$.

First we compute $G^{-1}(x)$

Case 1: q and s are even.

$$G^{-1}(x) = \begin{cases} \frac{x-q}{p} & \text{if } x \text{ is even} \\ \frac{x-s}{r} & \text{if } x \text{ is odd} \end{cases}$$

Case 2: q and s are odd.

$$G^{-1}(x) = \begin{cases} \frac{x-s}{r} & \text{if } x \text{ is even} \\ \frac{x-q}{p} & \text{if } x \text{ is odd} \end{cases}$$

Now we compute $(G \circ T \circ G^{-1})(x)$.

Case 1: q and s are even.

$$(T \circ G^{-1})(x) = \begin{cases} \frac{x-q}{2p} & \text{if } x \text{ is even} \\ \frac{3x-3s+r}{2r} & \text{if } x \text{ is odd} \end{cases}$$

Case 2: q and s are odd.

$$(T \circ G^{-1})(x) = \begin{cases} \frac{3x-3s+r}{2r} & \text{if } x \text{ is even} \\ \frac{x-q}{2p} & \text{if } x \text{ is odd} \end{cases}$$

At this point the function becomes rather complicated, so we will employ a slight abuse of notation to express it more compactly.

$$(G \circ T \circ G^{-1})(x) = \begin{cases} \frac{x+q}{2} & \text{if } x \text{ is even}; q, s \text{ are even}; \frac{x-q}{2} \text{ is even} \\ \frac{rx-rq+2ps}{2p} & \text{if } x \text{ is even}; q, s \text{ are even}; \frac{x-q}{2} \text{ is odd} \\ \frac{x+q}{2} & \text{if } x \text{ is odd}; q, s \text{ are odd}; \frac{x-q}{2} \text{ is even} \\ \frac{rx-rq+2ps}{2p} & \text{if } x \text{ is odd}; q, s \text{ are odd}; \frac{x-q}{2} \text{ is odd} \\ \frac{3px-3ps+pr+2qr}{2r} & \text{if } x \text{ is odd}; q, s \text{ are even}; \frac{3x-3s+r}{2} \text{ is even} \\ \frac{3x-s+r}{2} & \text{if } x \text{ is odd}; q, s \text{ are even}; \frac{3x-3s+r}{2} \text{ is odd} \\ \frac{3px-3ps+pr+2qr}{2r} & \text{if } x \text{ is even}; q, s \text{ are odd}; \frac{3x-3s+r}{2} \text{ is even} \\ \frac{3x-s+r}{2} & \text{if } x \text{ is even}; q, s \text{ are odd}; \frac{3x-3s+r}{2} \text{ is odd} \end{cases}$$

In particular, if we let $r \equiv 1 \bmod 4$ and $q, s \equiv 0 \bmod 4$ then the function may be simplified to

$$f(x) = \begin{cases} \frac{x+q}{2} & \text{if } x \equiv 0 \bmod 4 \\ \frac{3px-3ps+pr+2qr}{2r} & \text{if } x \equiv 1 \bmod 4 \\ \frac{rx-rq+2ps}{2p} & \text{if } x \equiv 2 \bmod 4 \\ \frac{3x-s+r}{2} & \text{if } x \equiv 3 \bmod 4. \end{cases}$$

Now we will show that this function is not, in general, a member of $\mathcal{F}$. Let $q = 0, s = 4$, and $p = r = 1$. Then for all even x

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \equiv 0 \bmod 4 \\ \frac{x+8}{2} & \text{if } x \equiv 2 \bmod 4. \end{cases}$$

If $f \in \mathcal{F}$ then $f = f_{a,b,c,d}$ for some $a, b, c, d \in \mathbb{Z}_2$ where a, c and d are odd and b is even. Now, if $x = 0$ we have $f(0) = 0$ so $f_{a,b,c,d}(0) = \frac{b}{2} = 0$ so $b = 0$. If $x = 4$ then we have $f(4) = 2$ so $f_{a,b,c,d}(4) = \frac{4a}{2} = 2$ so $a = 1$. Now $f(2) = \frac{2+8}{2} = 5$ but $f_{a,b,c,d}(2) = \frac{2}{2} = 1$. So $f(2) \neq f_{a,b,c,d}(2)$ therefore f is not a member of $\mathcal{F}$. Thus not all functions which are topologically conjugate to T by a piecewise linear map are members of $\mathcal{F}$.

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